

MARKSCHEME

May 2007

FURTHER MATHEMATICS

Standard Level

Paper 2

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**; working must be seen.
- (M)** Marks awarded for **Method**; may be implied by **correct** subsequent working.
- A** Marks awarded for an **Answer** or for **Accuracy**: often dependent on preceding **M** marks.
- (A)** Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.
- R** Marks awarded for clear **Reasoning**.
- N** Marks awarded for **correct** answers if **no** working shown.
- AG** Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations **MI**, **AI**, etc.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award **M0** followed by **AI**, as **A** mark(s) are often dependent on the preceding **M** mark.
- Where **M** and **A** marks are noted on the same line, e.g. **MIAI**, this usually means **MI** for an **attempt** to use an appropriate method (e.g. substitution into a formula) and **AI** for using the **correct** values.
- Where the markscheme specifies (**M2**), **N3**, etc, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 N marks

*Award N marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of **N** and other marks.
- There may be fewer **N** marks available than the total of **M**, **A** and **R** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks.
- If the error leads to an inappropriate value (e.g. $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent A** marks can be awarded, but **M** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). Apply a **MR** penalty of 1 mark to that question. Award the marks as usual and then write $-1(\mathbf{MR})$ next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks (*d*)

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled (**d**) and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for part-questions are indicated by **EITHER . . . OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates may not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x - 3)$, the markscheme gives:

$$f'(x) = 2\cos(5x - 3) \cdot 5 = 10\cos(5x - 3) \quad \text{AI}$$

Award **AI** for $2\cos(5x - 3) \cdot 5$, even if $10\cos(5x - 3)$ is not seen.

10 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

- **Rounding errors:** only applies to final answers not to intermediate steps.
- **Level of accuracy:** when this is not specified in the question the general rule applies: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Candidates should be penalized **once only IN THE PAPER** for an accuracy error (**AP**). Award the marks as usual then write (**AP**) against the answer. On the **front** cover write $-1(\text{AP})$. Deduct 1 mark from the total for the paper, not the question.

- If a final correct answer is incorrectly rounded, apply the **AP**.
- If the level of accuracy is not specified in the question, apply the **AP** for correct answers not given to three significant figures.

If there is no working shown, and answers are given to the correct two significant figures, apply the **AP**. However, do **not** accept answers to one significant figure without working.

11 Examples

Exemplar material is available under examiner training on examnet. Please refer to this material before you start marking, and when you have any queries. Please also feel free to contact your Team Leader if you need further advice.

1. (a) (i) Because the graph has vertices of odd degree. **R2**
 (ii) We are looking for 2 disjoint sets. **(M1)**

Put A in Set 1. Then B and D have to go in Set 2. This means that E and C have to go in Set 1. Therefore the disjoint sets are {B, D} and {A, C, E}. All the edges join a vertex from one set to a vertex in the other set.
 The graph is bipartite.

R2
A1

(iii)
$$M = \begin{pmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \end{pmatrix}$$
 A2

We require the (1, 3) or (3, 1) element of M^4 . **M1M1**
 Using a GDC, the number of walks of length 4 is 36. **A2**

[12 marks]

(b)	Step	Vertices labelled	Working values	
	1	P	P(0), Q-8, U-4	M1
	2	P, U	P(0), U(4), Q-7, T-11	A2
	3	P, U, Q	P(0), U(4), Q(7), R-14, T-9	A2
	4	P, U, Q, T	P(0), U(4), Q(7), T(9), R-12, S-18	A1
	5	P, U, Q, T, R	P(0), U(4), Q(7), T(9), R(12), S-18	A1
	6	P, U, Q, T, R, S	P(0), U(4), Q(7), T(9), R(12), S(18)	A1

The length of the shortest path is 18. **A2**

EITHER

P U Q T R S **A2**

OR

P U Q T S **A2**

[12 marks]

Total [24 marks]

2. (a) (i) H_0 : The data are $B(4, p)$; H_1 : The data are not $B(4, p)$ **A1**

(ii) Mean = $\frac{1 \times 32 + \dots + 4 \times 14}{100}$ **M1A1**

= 1.8 **A1**

$4\hat{p} = 1.8 \Rightarrow \hat{p} = 0.45$ **M1A1**

(iii) The expected frequencies are

Value of x	Obs freq	Exp freq
0	17	9.15
1	32	29.95
2	19	36.75
3	18	20.05
4	14	4.10

A1A1A1A1A1

The last two classes must be combined because the expected frequency for $x=4$ is less than 5.

R1

$\chi^2 = \frac{17^2}{9.15} + \frac{32^2}{29.95} + \frac{19^2}{36.75} + \frac{32^2}{24.15} - 100$ **M2**

= 18.0 **A2**

DF = 2 **(A1)**

Critical value = 9.21 **A1**

We conclude, at the 1 % significance level, that X does not fit a binomial model.

R1

[19 marks]

Special case: award the following marks to candidates who do not combine classes.

$\chi^2 = \frac{17^2}{9.15} + \frac{32^2}{29.95} + \frac{19^2}{36.75} + \frac{18^2}{20.05} + \frac{14^2}{4.1} - 100$ **M2**

= 39.6 **A0**

DF = 3 **(A1)**

Critical value = 11.345 **A1**

We conclude, at the 1 % significance level, that X does not fit a binomial model.

R1

continued ...

Question 2 continued

- (b) (i) Under H_0 , the distribution of \bar{y} is $N(500, 6.4)$. (AI)
Significance level = $P(\bar{y} < 495 \text{ or } > 505 | H_0)$ (M2)
 $= 2 \times 0.02405$ (AI)
 $= 0.0481$ AI N5

Note: Using tables, answer is 0.0478.

- (ii) The distribution of \bar{y} is now $N(503, 6.4)$. (AI)
 $P(\text{Type II error}) = P(495 < \bar{y} < 505)$ (M1)
 $= 0.785$ AI N3

Note: Using tables, answer is 0.784.

[8 marks]

Total [27 marks]

3. (a) Since the lengths of the two tangents from a point to a circle are equal **(M1)**
 $AF = AE, BF = BD, CD = CE$ **A1**

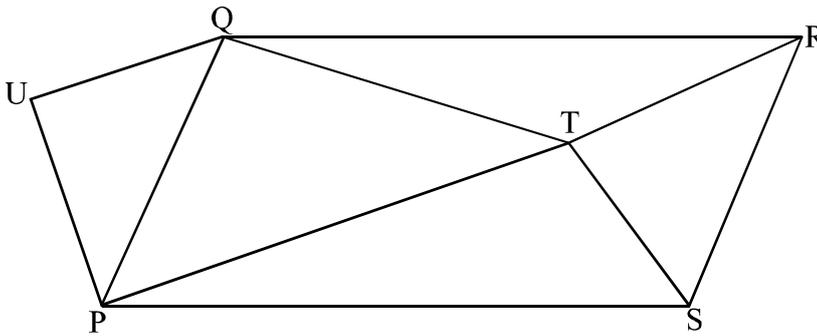
Consider

$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1 \text{ (signed lengths are not relevant here)} \quad \text{M2A2}$$

It follows by the converse to Ceva's Theorem that $[AD], [BE], [CF]$ are concurrent. **R2**

[8 marks]

(b)



Draw the ΔPQU congruent to ΔSRT (or translate ΔSRT to form ΔPQU). **R2**

Since $\hat{P}UQ = \hat{S}TR$, and $\hat{S}TR + \hat{P}TQ = 180^\circ$ it follows that **R2**

$$\hat{P}UQ + \hat{P}TQ = 180^\circ \quad \text{R1}$$

The quadrilateral $PUQT$ is therefore cyclic **R2**

Using Ptolemy's Theorem, **M2**

$$UQ \times PT + PU \times QT = PQ \times UT \quad \text{A2}$$

Since $UQ = TR, PU = ST$ and $UT = QR$ **R2**

$$\text{Then } PT \times TR + ST \times TQ = PQ \times QR \quad \text{AG}$$

[13 marks]

Total [21 marks]

4. (a) (i) $f^{(n)}(x) = \frac{e^x + (-1)^n e^{-x}}{2}$ (MI)AI

(ii) Coefficient of $x^n = \frac{f^{(n)}(0)}{n!}$ (MI)

$$= \frac{1 + (-1)^n}{2n!}$$
(AI)

$$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \dots$$
AI

(iii) Putting $x = \frac{1}{2}$ MI

$$f(0.5) = 1 + \frac{1}{8} + \frac{1}{16 \times 24} = \frac{433}{384}$$
(MI)AI

(iv) Lagrange error term $= \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$ MI

$$= \frac{f^{(5)}(c)}{120} \times \left(\frac{1}{2}\right)^5$$
AI

$f^{(5)}(c)$ is an increasing function because – any valid reason, e.g. plotted a graph, positive derivative, increasing function minus a decreasing function, so this is maximized when $x = 0.5$. RI

Therefore upper bound $= \frac{(e^{0.5} - e^{-0.5})}{2 \times 120} \times \left(\frac{1}{2}\right)^5$ MI
 $= 0.000136$ AI

[13 marks]

(b) We consider $\int_1^\infty \frac{\ln x}{x^2} dx = \int_1^\infty \ln x dx \left(-\frac{1}{x}\right)$ MIAI

$$= \left[-\frac{\ln x}{x}\right]_1^\infty + \int_1^\infty \frac{1}{x^2} dx$$
AIAI

$$= \left[-\frac{\ln x}{x}\right]_1^\infty - \left[\frac{1}{x}\right]_1^\infty$$
AI

Now $\lim_{x \rightarrow \infty} \left(\frac{1}{x}\right) = 0$ RI

$$\lim_{x \rightarrow \infty} \left(\frac{\ln x}{x}\right) = \lim_{x \rightarrow \infty} \left(\frac{1}{x}\right) = 0$$
MIAI

The integral is convergent with value 1 and so therefore is the series. RI

[9 marks]

Total [22 marks]

5. (a) (i) $3^x \equiv 3^x \pmod{10} \Rightarrow xRx$ so R is reflexive. **R1**
 $xRy \Rightarrow 3^x \equiv 3^y \pmod{10} \Rightarrow 3^y \equiv 3^x \pmod{10} \Rightarrow yRx$
 so R is symmetric. **R2**
 xRy and $yRz \Rightarrow 3^x - 3^y = 10M$ and $3^y - 3^z = 10N$
 Adding, $3^x - 3^z = 10(M + N) \Rightarrow 3^x \equiv 3^z \pmod{10}$ hence transitive. **R2**
- (ii) Consider $3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81, 3^5 = 243, \dots$ **(M2)**
 It is evident from this sequence that there are 4 equivalence classes,
 $1, 5, 9, \dots$ **A1**
 $2, 6, 10, \dots$ **A1**
 $3, 7, 11, \dots$ **A1**
 $4, 8, 12, \dots$ **A1**
[11 marks]
- (b) (i) Consider $(a + b\sqrt{3})(c + d\sqrt{3}) = (ac + 3bd) + (bc + ad)\sqrt{3}$ **M1A1**
 This establishes closure since products of rational numbers are rational. **R1**
 Since if a and b are not both zero and c and d are not both zero, it follows that $ac + 3bd$ and $bc + ad$ are not both zero. **R1**
 The identity is $1(\in S)$. **R1**
- Consider $(a + b\sqrt{3})^{-1} = \frac{1}{a + b\sqrt{3}}$ **M1A1**

$$= \frac{1}{a + b\sqrt{3}} \times \frac{a - b\sqrt{3}}{a - b\sqrt{3}}$$
 A1

$$= \frac{a}{(a^2 - 3b^2)} - \frac{b}{(a^2 - 3b^2)}\sqrt{3}$$
 A1
- This inverse $\in S$ because $(a^2 - 3b^2)$ cannot equal zero since a and b cannot both be zero **R1**
 and $(a^2 - 3b^2) = 0$ would require $\frac{a}{b} = \pm\sqrt{3}$ which is impossible
 because a rational number cannot equal $\sqrt{3}$. **R2**
 Finally, multiplication of numbers is associative. **R1**
- (ii) If a and b are both real numbers, $a + b\sqrt{3}$ would have no inverse if $a^2 = 3b^2$. **R2**
[15 marks]
Total [26 marks]
-